SIMULATION OF REDUCTION ORDER MODEL LINEAR PARAMETER VARYING (*LPV*) SYSTEMS USING MATLAB SOFTWARE

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Abstract

The use of optimal control techniques on high-order systems produce high-order controller. Therefore, an approximation from high-order system to low-order system is needed. The approximation is known as reduction model. In this paper we study one of reduction model of Linear Parameter Varying (*LPV*) i.e. *method balanced truncation*. The procedure of this method can be stated as follows: *first,* the quadratic stable (*Q*-stable) is shown for given a high-order of *LPV*. *Second,* the state space realization of the high-order *LPV* plant is transformed to the balanced realization. *Third,* the balanced realizations are truncated to obtain the reduced-order plant. *Fourth,* the reduced-order plant is shown similar properties with the high-order plant. Finally, the simulation is carried out for a missile autopilot by using *LMI Control toolbox and Robust Control toolbox* in MATLAB software. From the simulation results we obtain that the reduction system has similar properties with the high-order system, i.e. *Q*-stable and balanced.

Keywords: *LPV system, Q-stable, LPV Balanced Truncation, LMI Control toolbox and Robust Control toolbox.*

1. Introduction

The approximation of high-order plant and controller models by models of lower-order is an integral part of control system design. The model reduction was often based on physical intuition, for example mechanical engineers remove high- frequency vibration modes from models of aircraft wings, turbine shafts and flexible structures. It may also be possible to replace high-order controllers by low order approximations with little sacrifice in performance [4].

Simple linear models or controllers are normally preferred over complex one in control system design for some obvious reason: They are much easier to implement and more reliable as they are fewer things to go wrong in hardware or bugs to fix in software. In this paper we consider the problem of reducing the order of a linear multivariable dynamical system. Two well known Linear Time Invariants (*LTI*) approximation methods - Optimal Hankel Norm approximation and Balanced Truncation are extended to the Linear Parameter Varying (*LPV*) framework. However, we shall study only one of them: the balanced truncation method. The main advantage of this method is that it is simple and performs fairly well [3].

MATLAB software has a rich collection of functions immediately useful to the control engineer or system theorist. Eigenvalues, root-finding and matrix inversion are just a few examples of MATLAB's important numerical tools. More generally, MATLAB's linear algebra, matrix computation and numerical analysis capabilities provide a reliable foundation for control system engineering as well as many other disciplines. In this paper, we use the control system toolbox, particularly LMI control toolbox and robust control toolbox, to provide state-of-the-art tools for the LMI-based analysis and design of robust control systems [5, 7].

2. Literature Survey

Linear Parameter Varying (*LPV*) systems are a special class of time varying systems where the time dependence enters the state equation through one possibly more exogenous parameters [1]. Consider a system which has a state space realization given by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) &= C(\rho(t))x(t) + D(\rho(t))u(t) \end{aligned} \tag{1}$$

where $A: R^s \to R^{nxn}$, $B: R^s \to R^{nxm}$, $C: R^s \to R^{pxn}$, and $D: R^s \to R^{pxm}$ are continuous functions of the parameter vector $\rho \in R^s$. Note that there is no assumption that the parameter dependence exhibited by the state space matrices is linear. The state-space matrices of an *LPV* system in equation (1) as follow:

$$P_n(\rho(t)) = \left[\begin{array}{c|c} A(\rho(t)) & B(\rho(t)) \\ \hline C(\rho(t)) & D(\rho(t)) \end{array}\right] \tag{2}$$

Definition 1: Define the set of feasible parameter trajectories F_ρ to be a subset of all piecewise continuous functions $C^0 : R^+ \to R^s$, according to :

$$F_\rho \stackrel{\Delta}{=} \{\rho(t) : R^+ \to R^s,\ \rho_{i_{\min}} \le \rho_i \le \rho_{i_{\max}},\ i = 1, 2, \ldots, s\}$$

Continuity of the state-space matrices implies bounded ness on compact subsets of R^s and this ensures that for each $\rho(t) \in F_\rho$ the state transition matrix, denoted $\Phi_\rho(t, \tau)$ is unique and continuous. For this class of systems we define the notion of quadratic stability [1, 2, 6].

Definition 2 : The *LPV* system P_n with state-pace matrices given by equation (1) is quadratically stable (***Q*-stable**) if there exists a real positive-definite matrix $X = X^T > 0$ such that

$$A^T(\rho(t))\,X + X\,A(\rho(t)) < 0, \qquad \forall\ \rho(t) \in F_\rho. \tag{3}$$

Because $A(\rho(t))$ is continuous function of parameter $\rho(t) \in F_\rho$ compact, therefore the above definition shown that left equation is negative-definite uniform, that there exists scalar $\delta > 0$, such that :

$$A^T(\rho(t))\,X + X\,A(\rho(t)) \le -\delta I_n, \qquad \forall\ \rho(t) \in F_\rho.$$

In another word, $P_n(\rho(t))$ is a state-space realization of a *LPV* system ***Q*-stable** if only if there exists a real positive-definite matrix $P = P^T > 0$ and $Q = Q^T > 0$ such that:

$$A^T(\rho(t))Q + QA(\rho(t)) + C^T(\rho(t))C(\rho(t)) < 0, \ \forall \rho(t) \in F_\rho. \tag{4}$$

and

$$A(\rho(t))P + PA^T(\rho(t)) + B(\rho(t))B^T(\rho(t)) < 0, \ \forall \rho(t) \in F_\rho. \tag{5}$$

Henceforth we shall refer to a Q satisfying equation (4) as a parameter-varying observability Gramian and a P satisfying equation (5) as a parameter-varying controllability Gramian. Whereas $\rho(t)$ is written as ρ.

Definition 3 : (Induced L_2 Gain)
Given a ***Q*-stable** *LPV* system P_n, with zero initial condition, the Induced L_2 gain is defined as :

$$\|P_n(\rho)\|_{i,2} \overset{\Delta}{=} \sup_{\rho(t)\in F_\rho} \sup_{F_\rho \in L_2} \frac{\|P_n(\rho)u\|_2}{\|u\|_2} \tag{6}$$

Lemma 1 : (Quadratic Performance)
Given a continuous state-space realization of *LPV* system P_n, with a scalar $\gamma > 0$. If there exists an $X \in R^{nxn}$, $X = X^T > 0$ such that for all $\rho(t) \in F_\rho$:

$$\begin{bmatrix} A^T(\rho)X + XA(\rho) & XB(\rho) & \gamma^{-1}C^T(\rho) \\ B^T(\rho)X & -I & \gamma^{-1}D^T(\rho) \\ \gamma^{-1}C(\rho) & \gamma^{-1}D(\rho) & -I \end{bmatrix} < 0 \tag{7}$$

that:

1. $P_n(\rho)$ is Q-stable on F_ρ.
2. There exists a $\beta < \gamma$ such that $\| P_n(\rho)\|_{i,2} \leq \beta$.

A difficulty with determining quadratic performance using Lemma 1 is the infinite number of constraints which must be satisfied. However, by making the restrictions that

(1). The state-space matrices $A(\rho)$, $B(\rho)$, $C(\rho)$ and $D(\rho)$ depend affinely on ρ, i.e. :

$$A(\rho) = A_0 + \rho_1 A_1 + \ldots + \rho_s A_s;$$
$$B(\rho) = B_0 + \rho_1 B_1 + \ldots + \rho_s B_s;$$
$$C(\rho) = C_0 + \rho_1 C_1 + \ldots + \rho_s C_s;$$
$$D(\rho) = D_0 + \rho_1 D_1 + \ldots + \rho_s D_s.$$

Or, we can write as :

$$S(\rho) = S_0 + \rho_1 S_1 + \ldots + \rho_s S_s.$$

with $S(\rho) = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix}$

(2). That ρ varies in a convex polytope with a finite number of vertices $\{\rho_1, \rho_2, \ldots, \rho_N\}$.

Hence to determine performance for a system satisfying (1) and (2) it is sufficient to satisfy a finite number of constraints. Given a convex decomposition of the current parameter value $\rho = \alpha_1 \rho_1 + \alpha_2 \rho_2 + \alpha_3 \rho_3 + \ldots + \alpha_N \rho_N$, $\alpha_i > 0$, $\sum_{i=1}^{N} \alpha_i = 1$ then the state-space matrices of an affine *LPV* model are determined by

$$\begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A(\rho_i) & B(\rho_i) \\ C(\rho_i) & D(\rho_i) \end{bmatrix}.$$

2. Material and Method

Balancing of *LPV* systems

Definition 3 : Given an n – state quadratically stable *LPV* systems P_n, a parameter varying observability Gramian Q satisfying equation (4) and a parameter varying controllability Gramian P satisfying equation (5), define $\sigma_i = \sqrt{\lambda_i(QP)}$, $i = 1, 2, \ldots, n$ where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ to be P_n singular values and eigen values P_n.

Lemma 2: Given a continuous state-space realization of an *LPV* system P_n, an observability Gramian Q satisfying equation (4) and a controllability Gramian P satisfying equation (5) and a constant state transformation matrix T then

$$\tilde{P} = TPT^T$$
$$\tilde{Q} = T^{-T}QT^{-1}$$

are parameter varying observability and controllability Gramians for the transformed system respectively.

Proposition: Given a continuous state-space realization of an *LPV* system P_n, an observability Gramian Q satisfying equation (4) and a controllability Gramian P satisfying equation (5), then it is possible to find a constant state transformation matrix T such that the transformed Gramian $\tilde{P} = \tilde{Q} = \Sigma$. Σ is diagonal matrix which has P_n singular values arranged along its diagonal in descending order

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$

Definition 3: Given a continuous state-space realization of an *LPV* system P_n and a balancing state transformation matrix T such that $\tilde{P} = \tilde{Q} = \Sigma$, define the balanced parameter varying realization as follows

$$P_n(\rho) \stackrel{\Delta}{=} \left[\begin{array}{c|c} TA(\rho)T^{-1} & TB(\rho) \\ \hline C(\rho)T^{-1} & D(\rho) \end{array}\right]$$

***LPV* Balanced Truncation**

Lemma 3: Assume P_n is an n – state, quadratically stable, balanced *LPV* system partitioned as follows

$$P_n(\rho) \stackrel{\Delta}{=} \left[\begin{array}{cc|c} A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ \hline C_1(\rho) & C_2(\rho) & D(\rho) \end{array}\right]$$

Where : $A_{11} \in R^{rxr}$, $A_{12} \in R^{rx(n-r)}$, $A_{21} \in R^{(n-r)xr}$, $A_{22} \in R^{(n-r)x(n-r)}$, $B_1 \in R^{rxm}$, $B_2 \in R^{(n-r)xm}$, $C_1 \in R^{pxr}$, $C_2 \in R^{px(n-r)}$, $D \in R^{pxm}$.

With $P = Q = \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ and Σ_1 = diag (σ_1, σ_2, …, σ_r), Σ_2 = diag (σ_{r+1}, σ_{r+2}, …, σ_n), $\sigma_r > \sigma_{r+1}$ then :

$$P_r(\rho) \stackrel{\Delta}{=} \left[\begin{array}{c|c} A_{11}(\rho) & B_1(\rho) \\ \hline C_1(\rho) & D(\rho) \end{array}\right]$$

is an r-state, quadratically stable, balanced approximation to P_n (ρ) and the reduction error :

$$\| P_n(\rho) - P_r(\rho) \|_{i,2} < 2 \sum_{j=r+1}^{n} \sigma_j \cdot$$

Since balanced truncation of *LTI* systems produces zero error at infinite frequency because it does not influence the system's D matrix, but from control perspective we would like the approximation error to be a small at low and intermediate frequencies. It has been observed that in general case where more than one state is truncated the maximum approximation error resulting from balanced truncation occurs at low frequency. This has led to the development of several frequencies weighted approximation schemes which can be used to improve the approximant for control purposes. Of course parameter varying systems do not have a frequency response but we can still draw on intuition gained from *LTI* systems in order to improve the approximation for the purpose of control design. It can be seen in the following chapter.

3. Simulation and Discussion

In this example we examine reduced order controller synthesis for pitch axis control of a missile. The dynamic of the missile under consideration vary greatly as a function of speed (v), altitude (H) and angle-of-attack (α), hence a single *LTI* model and controller is insufficient for effective control. Here we'll consider an *LPV* model and controller. The model we will use has previously been studied by Gahinet et al. [10]. For the synthesis of *LPV* controllers of state-dimension equal to the weighted model. The parameter dependent model is given by,

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_f$$

$$\begin{bmatrix} a_{zv} \\ q \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix}$$

where Z_α and M_α are aerodynamic co-effficients which depend on v, H and α. The input system is the fin deflection (δ_f). The two states are angle-of-attack (α) and pitch rate (q) respectively. The two outputs are the (*normalized*) vertical acceleration (a_{zv}) pitch rate (q) respectively.

We assume that v, H and α may all be measured in real time and vary over the following ranges, i.e.:

$v \in [0.5 \;\; 4]$ Mach, $H \in [0 \;\; 1800]$m, and $\alpha \in [0 \;\; 40]$ degree.

The combination of v, H and α imply that $Z_\alpha \in [0.5\ \ 4]$, and $M_\alpha \in [0\ \ 106]$. The state-space realization of systems $P_\rho = \left[\begin{array}{cc|c} -Z_\alpha & 1 & 0 \\ -M_\alpha & 0 & 1 \\ \hline -1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

Note that Z_α and M_α enter the state-space matrices in an affine way, and may be considered as representing a convex polytope with four vertices.

The objective we wish to achieve is for the step response of a_{zv} to have a settling time of less than half a second for all variations of Z_α and M_α in the above range.

Given the performance weights,

$$W_1 = \frac{2.01}{s+0.201} \quad \text{and} \quad W_2 = \frac{96.78s^3 + 0.29s^2 + 0.0003s - 0.0039}{s^3 + 1.12\times10^5 s^2 + 1.05\times10^8 s + 1.07\times10^{11}}.$$

The performance specification correspond to designing a controller which achieves,

$$\left\| \begin{matrix} W_1(1+P_\rho K_\rho)^{-1} \\ W_2 P_\rho (1+P_\rho K_\rho)^{-1} \end{matrix} \right\|_\infty < 1, (\gamma = 1)$$

From the above problem, can be done as follow:

- Reducing all ***generalized plant*** by Balanced Truncation.
- Designing controllers from initial plant and reduced plant where step response of a_{zv} has settling time less than 0.6 second.

Solution:

To solve this problem, we use MATLAB software, mainly LMI Control Toolbox and Robust Control Toolbox.

Steps of solving phases:

1). Add weighted function in the plant, such that order – 6 generalized plant.

2). Show that generalized plant Q-stable.

3). Define matrix $A(\rho)$, $B(\rho)$, $C(\rho)$, $D(\rho)$. After obtaining these matrixes, the generalized plant is balanced. Therefore :

$$P = Q = \Sigma = \begin{bmatrix} 48.7991 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.0009 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6924 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2985 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2813 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2797 \end{bmatrix}$$

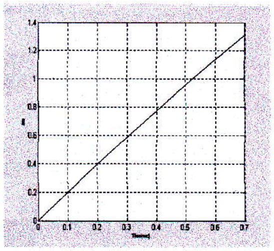
4). Because the generalized plant is Q-stable and balanced, so balanced truncation can be done. So that the order – 6 initial plant would truncate to become order - 5, order - 4, order - 3. The result of truncated can be seen in the following table.

Tabel.1: Reduction Error

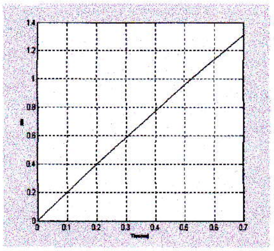
Order Plant	$k=5$	$k=4$	$k=3$
$\left\lVert P_w(\rho)-\hat{P}_w(\rho)\right\rVert_\infty$	0.559	0.560	0.814
$2trace\Sigma_2$	0.559	1.122	1.719

From the table shown above can be seen the gap between initial plant and reduced plant for continuously outcoming reduction error less than 2 times of singular values that is truncated ($2trace\Sigma_2 = 2\sum_{j=k+1}^{n}\sigma_j$).

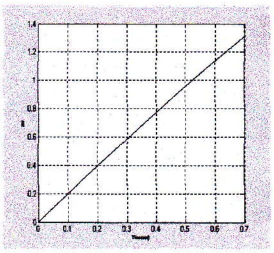
5). Next, we draw a step response graphic. From this graphic shown at picture.1, picture.2 and picture.3 can be seen that all graphic from reduction plant are lying over (overlapping) the same line of initial plant



Picture 1: Step response graphic for open loop order - 6 which is overlapping with open loop order-5

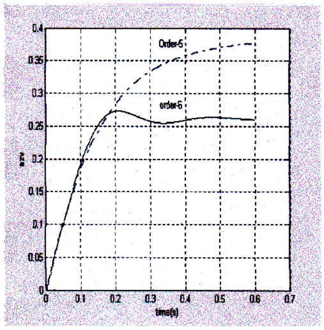


Picture 2: Step response graphic for open loop order - 6 which is overlapping with open loop order-4

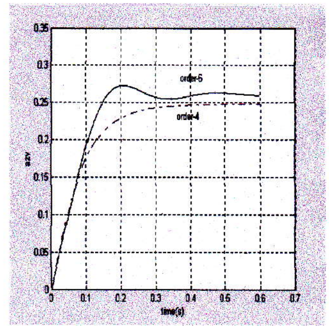


Picture 3: Step response graphic for open loop order - 6 which is overlapping with open loop order-3

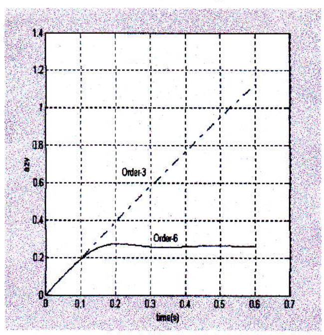
6). Finally, Designing controllers from initial plant and reduced plant where step response of a_{zv} has settling time less than 0.6 second. From picture.5 can be seen that time response closed loop system from initial plant order – 6 and reduced plant order – 4 which given time function, the controller has settling time less than 0.6 second. Conversely, another picture does not have settling time.



Picture 4: Step response graphic for open loop order - 6 which is overlapping with open loop order-5



Picture 5: Step response graphic for open loop order – 6 which is overlapping with open loop order-4



Picture 6: Step response graphic for open loop order - 6 which is overlapping with open loop order-3

4. Conclusion

In this paper we have shown that the *LTI* model approximation technique of balanced truncation may be extended to *LPV* framework, where the least upper bound from reduction error less than 2 times singular values reduced system. Furthermore, if initial plant are *Q*-stable and balanced, then reduced plant are *Q*-stable and balanced, too.

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